

Shear viscosity and spin sum rules in strongly interacting Fermi gases

Tilman Enss

Physik Department, Technische Universität München, 85747 Garching, Germany

Abstract. Fermi gases with short-range interactions are ubiquitous in ultracold atomic systems. In the absence of spin-flipping processes the number of atoms in each spin species is conserved separately, and we discuss the associated Ward identities. For contact interactions the spin conductivity spectral function $\sigma_s(\omega)$ has universal power-law tails at high frequency. We derive the spin f -sum rule and show that it is not affected by these tails in $d < 4$ dimensions. Likewise the shear viscosity spectral function $\eta(\omega)$ has universal tails; in contrast they modify the viscosity sum rule in a characteristic way.

1 Introduction

Condensed matter systems near a phase transition generally have universal low-energy properties, while the high-energy response depends on non-universal details of the microscopic interaction. Ultracold atoms provide an important exception: in dilute gases, where the range of the interaction r_e is much shorter than the mean particle spacing, also the high-energy properties are universal up to a cutoff energy \hbar^2/mr_e^2 set by the interaction range, which can be much larger than the Fermi or thermal energies [1]. The correlation functions have characteristic high-frequency and momentum tails which are controlled by the Tan contact density C [2, 3, 4, 5, 6]. This quantity measures the probability of finding two atoms of different species near each other. Together, two atoms can absorb a large kinetic energy and undergo a boost in opposite directions while conserving total momentum. Hence, the high-energy response of the system is proportional to the density C of such pairs.

In this work we look in particular at the response to a magnetic field gradient, the spin conductivity σ_s , which has recently been measured [7] and provides an example of quantum limited transport. Aspects of this are understood within kinetic theory [9, 10], while a recent strong-coupling Luttinger-Ward calculation [8] explains the spin diffusion quantitatively and predicts the full frequency dependence of the spin conductivity $\sigma_s(\omega)$. Furthermore, we consider the response to shear flow, the shear viscosity $\eta(\omega)$ [11, 12, 13, 14] in two and three spatial dimensions. The transport coefficients exhibit universal power-law tails at high frequencies, and we study how these tails affect the exact sum rules which link the frequency integrated response to the thermodynamic properties of the system [8, 13, 15, 16, 17, 18, 19]. The question of spin transport is connected with the conservation of the particle numbers N_σ of the spin species. For the case of a density-density interaction there are no spin-flipping processes and each N_σ is conserved separately. This implies spin-selective Ward identities for every spin species, which we will then use to derive the spin sum rule.

2 Model and symmetries

Consider a two-component Fermi gas with contact interaction which is described by the grand canonical Hamiltonian

$$H = \int d\mathbf{x} \sum_{\sigma} \psi_{\sigma}^{\dagger} \left(\partial_{\tau} - \frac{\nabla^2}{2m} - \mu_{\sigma} \right) \psi_{\sigma} + g_0 \psi_{\uparrow}^{\dagger} \psi_{\downarrow}^{\dagger} \psi_{\downarrow} \psi_{\uparrow} \quad (1)$$

with mass m , chemical potential μ_{σ} for spin species $\sigma = \uparrow, \downarrow$, and $\hbar \equiv 1$. At low energies s -wave scattering is allowed only between opposite spins by the Pauli principle. The contact interaction g_0 leads to ultraviolet divergences which need to be regularized [1].

The interacting Fermi gas (1) is invariant under a $U(1) \times U(1)$ symmetry corresponding to the separate conservation of \uparrow and \downarrow particle number. This is readily seen by coupling a different gauge field to each spin component [20]. In the absence of a magnetic field the symmetry is enlarged to $SU(2)$. The spin-selective particle number and current operators can be written as

$$\rho_{\sigma}(x) = \psi_{\sigma}^{\dagger}(x) \psi_{\sigma}(x) \quad (2)$$

$$j_{\sigma}^i(x) = -\frac{i}{2} \left(\psi_{\sigma}^{\dagger}(x) \partial_i \psi_{\sigma}(x) - \partial_i \psi_{\sigma}^{\dagger}(x) \psi_{\sigma}(x) \right) \quad (3)$$

where $x = (\mathbf{x}, t)$. These operators satisfy the continuity equation

$$\partial_t \rho_{\sigma}(x) + \partial_i j_{\sigma}^i(x) = \partial_{\mu} j_{\sigma}^{\mu}(x) = 0 \quad (4)$$

separately for each spin component with conserved particle numbers N_{\uparrow} and N_{\downarrow} . The bare current operator (3) acquires no interaction correction in the case of the density-density interaction (1) since $[\rho_{\sigma}(x), \rho_{\sigma'}(y)] = 0$ and $[H_{\text{int}} - \sum_{\sigma} \mu_{\sigma} N_{\sigma}, \rho_{\sigma'}(x)] = 0$ [21]. The continuity equation implies spin-selective Ward identities which connect the number ρ_{σ} and current j_{σ}^i response functions with the Green's functions. These have been derived by Behn [22],

$$\partial_{\mu} \langle T j_{\sigma}^{\mu}(x) \psi_{\sigma'}(y) \psi_{\sigma'}^{\dagger}(y') \rangle = \delta_{\sigma\sigma'} \langle T \psi_{\sigma}(y) \psi_{\sigma}^{\dagger}(y') \rangle [\delta(x-y) - \delta(x-y')] \quad (5)$$

or in momentum space

$$\begin{aligned} \omega \rho_{\sigma}(\mathbf{p}, \sigma', \epsilon; \mathbf{p} + \mathbf{q}, \sigma', \epsilon + \omega) - q_i j_{\sigma}^i(\mathbf{p}, \sigma', \epsilon; \mathbf{p} + \mathbf{q}, \sigma', \epsilon + \omega) \\ = \delta_{\sigma\sigma'} [G_{\sigma}(\mathbf{p} + \mathbf{q}, \epsilon + \omega) - G_{\sigma}(\mathbf{p}, \epsilon)] \end{aligned} \quad (6)$$

with Green's functions $G_{\sigma}^{-1}(\mathbf{p}, \omega) = -\omega + \varepsilon_{\mathbf{p}} - \mu_{\sigma} - \Sigma_{\sigma}(\mathbf{p}, \omega)$ and the free single-particle dispersion $\varepsilon_{\mathbf{p}} = \mathbf{p}^2/2m$. In particular, there is no response of the \downarrow Green's function to a $q_{\mu} j_{\uparrow}^{\mu}$ perturbation, which is not immediately obvious from looking at the perturbative contributions: indeed, Maki-Thompson and Aslamazov-Larkin vertex corrections [13] cancel exactly in this case. For $SU(2)$ invariant models there is an additional Ward identity for the σ^+ operator [22].

3 Spin f -sum rule

The correlation functions of number-current χ_{jn} and spin-current χ_{js} are defined as

$$\chi_{\text{jn/js}}(\mathbf{q}, \omega) = i \int_0^{\infty} dt d\mathbf{x} e^{i(\omega t - \mathbf{q} \cdot \mathbf{x})} \langle [j_{\text{n/s}}^z(\mathbf{x}, t), j_{\text{n/s}}^z(\mathbf{0}, 0)] \rangle \quad (7)$$

in terms of the number and spin current operators $j_{n/s}^i(x) = j_{\uparrow}^i(x) \pm j_{\downarrow}^i(x)$. The corresponding number and spin conductivities in the zero-momentum limit are defined in terms of the correlation function (7) as

$$\sigma_{n/s}(\omega) = \frac{\text{Im} \chi_{jn/js}(\mathbf{0}, \omega)}{\omega}. \quad (8)$$

A Kramers-Kronig transformation relates the frequency integral of $\sigma_{n/s}(\omega)$ to the current correlation function at zero frequency,

$$\int_{-\infty}^{\infty} \frac{d\omega}{\pi} \sigma_{n/s}(\omega) = \chi_{jn/js}(\mathbf{0}, \omega = 0), \quad (9)$$

which is real. The Kubo formula (7) can be expressed in terms of the fermionic Green's and response functions in the Matsubara formalism as [20, 13]

$$\chi_{js}(\mathbf{0}, 0) = -\frac{1}{\beta V} \sum_{\mathbf{p}\sigma\sigma'\epsilon_n} \frac{\tau_{\sigma\sigma'}^z p_z}{m} \times \tau_{\sigma\sigma'}^z j_{\sigma}^z(\mathbf{p}, \sigma', i\epsilon_n) = -\frac{1}{\beta V} \sum_{\mathbf{p}\sigma\epsilon_n} \frac{p_z}{m} j_{\sigma}^z(\mathbf{p}, \sigma, i\epsilon_n) \quad (10)$$

where $\tau_{\sigma\sigma'}^z p_z/m$ is the bare spin-current response vertex with Pauli matrix τ^z . This is multiplied with $j_{\sigma}^z(\mathbf{p}, \sigma', i\epsilon_n)$, the fully dressed current response function from Eq. (6) in the limit $\omega = 0, \mathbf{q} \rightarrow 0$. The Ward identity (6) for each spin component relates the current response function in the static limit $\omega = 0, \mathbf{q} \rightarrow 0$ to the Green's function,

$$j_{\sigma}^z(\mathbf{p}, \sigma, i\epsilon_n) = -\frac{\partial G_{\sigma}(p, i\epsilon_n)}{\partial p_z} = -\frac{p_z}{p} \frac{\partial G_{\sigma}(p, i\epsilon_n)}{\partial p}. \quad (11)$$

The Matsubara sum over the Green's function yields the momentum distribution $-\beta^{-1} \sum_{\epsilon_n} G(p, i\epsilon_n) = n_{\mathbf{p}\sigma}$ and one obtains

$$\chi_{js}(\mathbf{0}, 0) = -\frac{1}{V} \sum_{\mathbf{p}\sigma} \frac{p_z^2}{mp} \frac{\partial n_{p\sigma}}{\partial p} = \chi_{jn}(\mathbf{0}, 0). \quad (12)$$

The same result is obtained if one considers not the spin-current but the number-current with bare response vertex $\delta_{\sigma\sigma'} p_z/m$. The normalized integral over the solid angle Ω_d yields $\int d\Omega p_z^2/\Omega_d = p^2/d$ in $d \geq 2$ dimensions. Integration by parts over p then gives

$$\chi_{jn/js}(\mathbf{0}, 0) = -\sum_{\sigma} \int_0^{\Lambda} \frac{\Omega_d dp p^{d-1}}{(2\pi)^d} \frac{p}{md} \frac{\partial n_{p\sigma}}{\partial p} = \sum_{\mathbf{p}\sigma} \frac{n_{p\sigma}}{m} - \frac{1}{md} \frac{\Omega_d}{(2\pi)^d} \sum_{\sigma} p^d n_{p\sigma} \Big|_0^{\Lambda} \quad (13)$$

where we have explicitly written the ultraviolet momentum cutoff $\Lambda \sim 1/|r_e|$. The first term gives the density, while the second term depends on the cutoff. For zero-range interactions the momentum distribution at large momenta is proportional to the Tan contact density, $n_{p\sigma} = C/p^4$ as $p \rightarrow \infty$ in any dimension [2, 3, 4]. Hence, the cutoff term $C\Lambda^{d-4}$ vanishes for $\Lambda \rightarrow \infty$ ($r_e \rightarrow 0$) in any dimension $d < 4$. In combination with Eq. (9) this completes the derivation of the particle number and spin sum rule

$$\frac{1}{\pi} \int_{-\infty}^{\infty} d\omega \sigma_{n/s}(\omega) = \frac{n}{m} \quad (14)$$

with the total density $n = n_\uparrow + n_\downarrow$.

In the Galileian invariant model (1) the number current is proportional to momentum and cannot decay. This implies the conservation of the total number current,

$$[H, \int d\mathbf{x} j_n^i(x)] = 0, \quad (15)$$

and consequently the number conductivity has a sharp Drude peak at zero frequency,

$$\sigma_n(\omega) = \frac{n}{m} \pi \delta(\omega). \quad (16)$$

In contrast, the global spin current is not conserved because scattering transfers momentum between \uparrow and \downarrow particles,

$$[H, \int d\mathbf{x} j_s^i(x)] \neq 0, \quad (17)$$

and the spin conductivity $\sigma_s(\omega)$ has a finite and nontrivial response at $\omega > 0$. The spin conductivity in 3d has recently been computed in the Luttinger-Ward formalism [8]: there is a broad Drude peak at low frequencies, followed by a universal high-frequency tail

$$\sigma_s(\omega \rightarrow \infty) = \frac{C}{3\pi(m\omega)^{3/2}} \quad (3d) \quad (18)$$

in accordance with results from the operator product expansion [17]. Both in two and three dimensions the tail decays sufficiently fast for the frequency integral (14) to converge, so again the universal high-energy properties of the zero-range model do not affect the form of the spin f -sum rule in $d < 4$ dimensions.

4 Shear viscosity sum rule

The shear viscosity η measures the friction of a fluid subject to a shear flow of both spin species simultaneously (mass flow). The frequency dependent shear viscosity

$$\eta(\omega) = \frac{\text{Im } \chi_{xy}(\omega)}{\omega} \quad (19)$$

is defined via the stress correlation function

$$\chi_{xyxy}(\mathbf{q}, \omega) = i \int_0^\infty dt d\mathbf{x} e^{i(\omega t - \mathbf{q} \cdot \mathbf{x})} \langle [H_{xy}(\mathbf{x}, t), H_{xy}(\mathbf{0}, 0)] \rangle \quad (20)$$

at zero external momentum, $\mathbf{q} = 0$. The stress tensor operator has the off-diagonal components [21, 13]

$$H_{xy} = \sum_{\mathbf{p}} \frac{p_x p_y}{m} c_{\mathbf{p}-\mathbf{q}/2, \sigma}^\dagger c_{\mathbf{p}+\mathbf{q}/2, \sigma} \quad (21)$$

since the interaction correction vanishes in the zero-range limit [21, 13]. Again a Kramers-Kronig transformation relates the frequency integral of the viscosity to the stress correlation function at zero external frequency (static limit),

$$\int_{-\infty}^\infty \frac{d\omega}{\pi} \eta(\omega) = \chi_{xyxy}(\omega = 0). \quad (22)$$

In analogy with the spin case, the Kubo formula (20) is expressed in terms of the stress response function T_{xy} as [13]

$$\chi_{xyxy}(0) = -\frac{1}{\beta V} \sum_{\mathbf{p}\sigma\epsilon_n} \frac{p_x p_y}{m} T_{xy}(\mathbf{p}, i\epsilon_n). \quad (23)$$

In the static limit of external $\omega = 0$, $\mathbf{q} \rightarrow 0$ the stress response is determined by the Ward identity associated with momentum current conservation [23],

$$T_{xy}(\mathbf{p}, i\epsilon_n) = -p_x \frac{\partial G(p, i\epsilon_n)}{\partial p_y} = -\frac{p_x p_y}{p} \frac{\partial G(p, i\epsilon_n)}{\partial p} \quad (24)$$

and hence

$$\chi_{xyxy}(0) = -\frac{1}{V} \sum_{\mathbf{p}\sigma} \frac{p_x^2 p_y^2}{mp} \frac{\partial n_{p\sigma}}{\partial p}. \quad (25)$$

The normalized integral over the solid angle Ω_d yields $\int d\Omega_d p_x^2 p_y^2 / \Omega_d = p^4 / [d(d+2)]$ in $d \geq 2$ dimensions. Performing an integration by parts as in Eq. (13) relates the correlation function to the kinetic energy density $E_{\text{kin}} = \frac{1}{V} \sum_{\mathbf{p}\sigma} \varepsilon_p n_{p\sigma}$. The integrals are ultraviolet divergent and can be regularized by a momentum cutoff Λ ,

$$\chi_{xyxy}(0) = \frac{2}{d} E_{\text{kin}} - \frac{1}{md(d+2)} \frac{\Omega_d}{(2\pi)^d} \sum_{\sigma} p^{d+2} n_{p\sigma} \Big|_0^{\Lambda}. \quad (26)$$

Through the momentum distribution $n_{p\sigma} = C/p^4$ for $p \rightarrow \infty$ (see above) the cutoff term depends on the contact density C ,

$$\chi_{xyxy}(0) = \frac{2}{d} E_{\text{kin}} - \frac{\Omega_d}{(2\pi)^d} \frac{C \Lambda^{d-2}}{md(d+2)}. \quad (27)$$

The kinetic energy density E_{kin} can be re-written using the Tan relations for the internal energy density ε or the pressure P as [2, 3]

$$E_{\text{kin}} = \varepsilon + \frac{C}{4\pi m} \ln \frac{\omega_{\Lambda}}{\varepsilon_B} = P - \frac{C}{4\pi m} + \frac{C}{4\pi m} \ln \frac{\omega_{\Lambda}}{\varepsilon_B} \quad (2d) \quad (28)$$

$$E_{\text{kin}} = \varepsilon - \frac{C}{4\pi m} \left(\frac{1}{a} - \frac{2\Lambda}{\pi} \right) = \frac{3}{2} \left[P - \frac{C}{4\pi m a} + \frac{C \Lambda}{3\pi^2 m} \right] \quad (3d) \quad (29)$$

with the cutoff energy $\omega_{\Lambda} = 2\varepsilon_{\Lambda} = \Lambda^2/m$ and the two-particle binding energy ε_B . Then the stress correlation function including the cutoff term in Eq. (27) reads

$$\chi_{xyxy}(0) = P - \frac{3C}{8\pi m} + \frac{C}{4\pi m} \ln \frac{\omega_{\Lambda}}{\varepsilon_B} \quad (2d) \quad (30)$$

$$\chi_{xyxy}(0) = P - \frac{C}{4\pi m a} + \frac{4C\sqrt{m\omega_{\Lambda}}}{15\pi^2 m} \quad (3d). \quad (31)$$

The zero-range interaction leads to universal high-frequency tails $\eta(\omega) \sim C/(8m\omega)$ in 2d [17, 18] and $\eta(\omega) \sim C/(15\pi\sqrt{m\omega})$ in 3d [13, 16]. These tails have to be subtracted to make the frequency integral (22) convergent, and one obtains the shear viscosity sum rules [13, 16, 19]

$$\frac{2}{\pi} \int_0^{\infty} d\omega \left[\eta(\omega) - \frac{C}{8m\omega} \Theta(\omega - \varepsilon_B) \right] = P - \frac{3C}{8\pi m} = \varepsilon - \frac{C}{8\pi m} \quad (2d) \quad (32)$$

$$\frac{2}{\pi} \int_0^{\infty} d\omega \left[\eta(\omega) - \frac{C}{15\pi\sqrt{m\omega}} \right] = P - \frac{C}{4\pi m a} = \frac{2}{3}\varepsilon - \frac{C}{6\pi m a} \quad (3d). \quad (33)$$

The universal high-frequency behavior is most clearly seen if one looks near the quantum critical point at zero density and zero temperature [24,25]. The shear viscosity in this limit but with the same value of C as in the dense system has the form [19]

$$\eta_0(\omega) = \frac{C}{8m\omega} \left(1 - \frac{\varepsilon_B}{\omega}\right)^2 \Theta(\omega - \varepsilon_B) \quad (2d). \quad (34)$$

By subtracting $\eta_0(\omega)$ one arrives a low-energy sum rule which captures only the finite-density effects [19]

$$\frac{2}{\pi} \int_0^\infty d\omega [\eta(\omega) - \eta_0(\omega)] = P \quad (2d). \quad (35)$$

In conclusion, we have argued that zero-range interactions realized in ultracold atomic systems do not modify the spin f -sum rule but lead to characteristic contact terms in the shear viscosity.

References

1. I. Bloch, J. Dalibard, and W. Zwerger, Rev. Mod. Phys. **80**, 885 (2008).
2. S. Tan, Ann. Phys. (NY) **323**, 2971 (2008); S. Tan, Ann. Phys. (NY) **323**, 2952 (2008).
3. F. Werner and Y. Castin, arXiv:1001.0774; F. Werner and Y. Castin, arXiv:1204.3204.
4. M. Barth and W. Zwerger, Ann. Phys. (NY) **326**, 2544 (2011).
5. M. Valiente, N. T. Zinner, and K. Mølmer, Phys. Rev. A **84**, 063626 (2011).
6. E. Braaten, in *The BCS-BEC Crossover and the Unitary Fermi Gas*, edited by W. Zwerger (Springer, 2012), p. 193.
7. A. Sommer, M. Ku, G. Roati, and M. W. Zwierlein, Nature (London) **472**, 201 (2011).
8. T. Enss and R. Haussmann, arXiv:1207.3103.
9. G. M. Bruun, Phys. Rev. A **85**, 013636 (2012).
10. T. Enss, C. Küppersbusch, and L. Fritz, Phys. Rev. A **86**, 013617 (2012).
11. C. Cao, E. Elliott, J. Joseph, H. Wu, J. Petricka, T. Schäfer, and J. E. Thomas, Science **331**, 58 (2011).
12. P. Massignan, G. M. Bruun, and H. Smith, Phys. Rev. A **71**, 033607 (2005).
13. T. Enss, R. Haussmann, and W. Zwerger, Ann. Phys. (NY) **326**, 770 (2011).
14. C. Chafin and T. Schäfer, arXiv:1209.1006; P. Romatschke and R. E. Young, arXiv:1209.1604.
15. L. P. Pitaevskii and S. Stringari, *Bose-Einstein Condensation* (Oxford University Press, 2003).
16. E. Taylor and M. Randeria, Phys. Rev. A **81**, 053610 (2010).
17. J. Hofmann, Phys. Rev. A **84**, 043603 (2011).
18. W. D. Goldberger and Z. U. Khandker, Phys. Rev. A **85**, 013624 (2012).
19. E. Taylor and M. Randeria, arXiv:1205.1525.
20. A. A. Abrikosov, L. P. Gorkov, and I. E. Dzyaloshinski, *Methods of quantum field theory in statistical physics* (Dover, 1975).
21. Y. Nishida and D. T. Son, Phys. Rev. D **76**, 086004 (2007).
22. U. Behn, physica status solidi (b) **88**, 699 (1978).
23. A. M. Polyakov, JETP **30**, 1164 (1969) [Zh. Eksp. Teor. Fiz. **57**, 2144 (1969)].
24. P. Nikolić and S. Sachdev, Phys. Rev. A **75**, 033608 (2007).
25. T. Enss, Phys. Rev. A **86**, 013616 (2012).